# THREE DIMENSIONAL FIGURES

## Three Dimensional Figures

Sometimes before you make any purchases you may want to know for example, how much cloth you need to make a pillow cover. What about a cover for your mattress or sofa cushion? How much oil paint do you need to paint your drinking water tank?

What about the amount of cloth for the pocket covers of your radio, curtain, suit, gown, trousers, set of table clothes, etc.

Answers to such questions and of the kind leads you to think more carefully about the size of the surfaces (faces) to be covered or coated on the bodies at work. Perhaps you need to take some measurements on the surfaces.

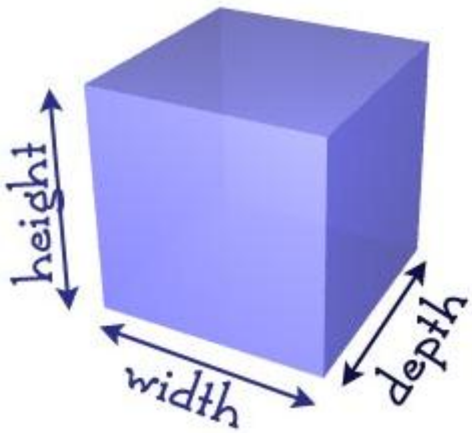
The knowledge of the surface areas of such bodies will enable you to choose or purchases the required amount without unnecessary wastage so as to minimize purchases costs too.

## Three Dimensional Figures

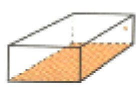
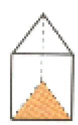
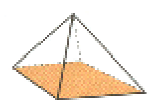
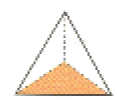
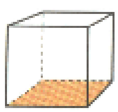
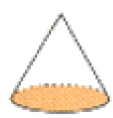
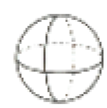
*Classify three dimensional figures*

***Three-dimensional objects*** are the ***solid shapes*** you see every day, like boxes, balls, coffee cups, and cans.

It is called **three-dimensional** or **3D** because there are three dimensions: *width, depth* and *height.*




-The following table shows examples of some common three dimensional figures

| | | | |
|---|---|---|---|
|  Rectangular Prism |  Triangular Prism |  Rectangular Pyramid |  Triangular Pyramid |
|  Cube |  Cylinder |  Cone |  Sphere |

## The Characteristics of Each Class

*List the characteristics of each class*

Here are some helpful vocabulary terms for solids:

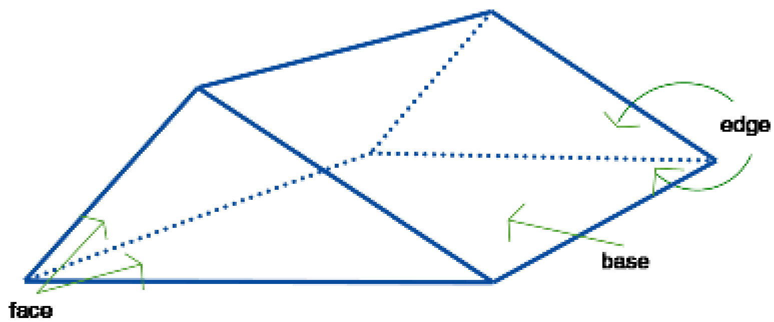
**Base**: Is the bottom surface of a solid object.

**Edge**: Is the intersection of two faces on a solid object. This is a line.

**Face**: Is a flat side of a 3-dimensional object.

**Prism**: Is a solid object with two congruent and parallel faces.

**Pyramid**: Is a solid object with a polygon for a base and triangles for sides.




## Construction of Three Dimensional Figures

### Three Dimensional Figures

*Construct three dimensional figures*

When drawing a three dimensional object it is important to show that it is not a drawing of a flat object. Are usually drawn on a two dimensional plane by making oblique drawings under certain rules as follows:

1. Paralled lines are drawn parallel.
2. Vertical lines are drawn up and down the page.
3. Hidden edges are drawn dotted.
4. Construction lines to guide the eyes are drawn thinly.

**Activity 1**

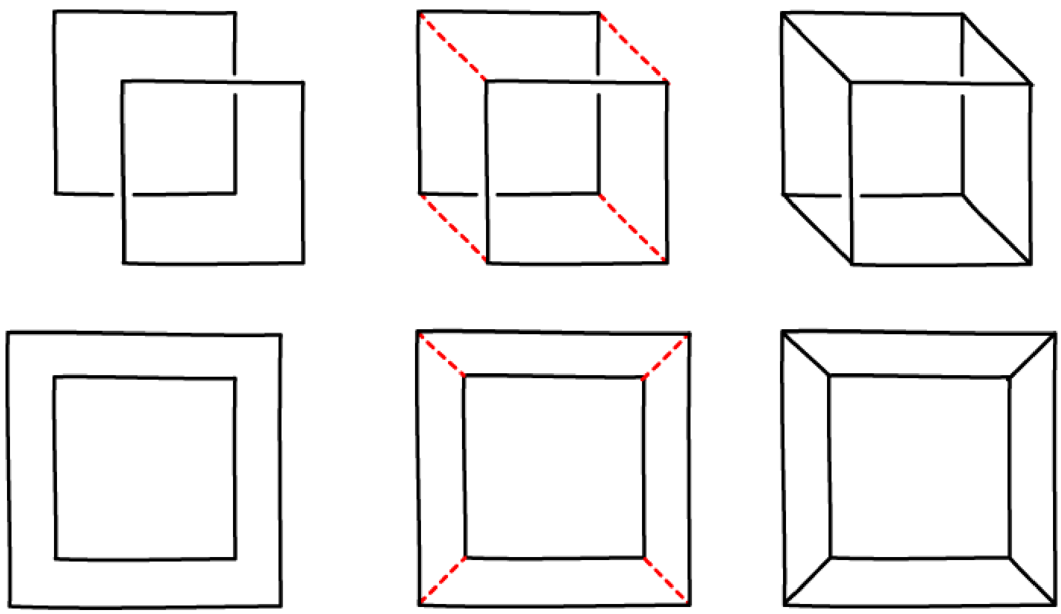
Construct three dimensional figures

## Sketching Three Dimensional Figures

### Three Dimensional Figures

*Sketch three dimensional figures*

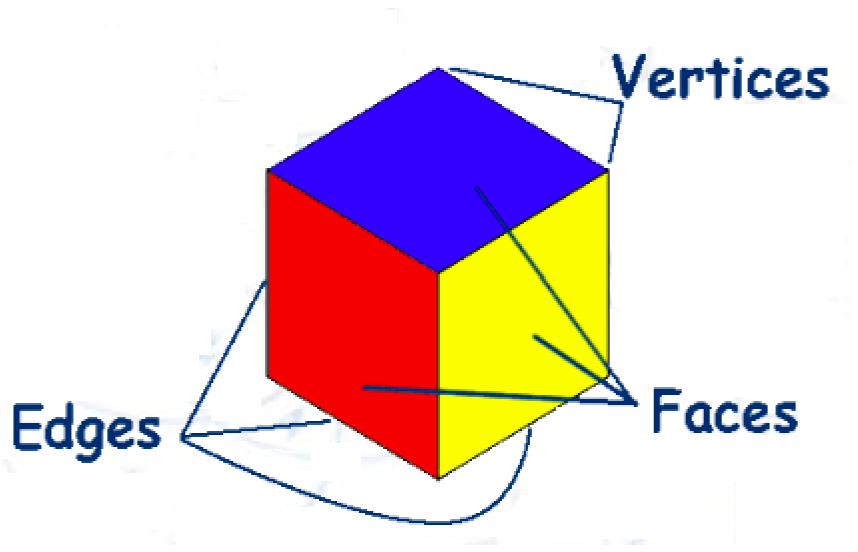
There are several ways of doing the drawing that corresponds to looking at the cube from different angles. The figure shows two ways of doing it.



## Properties of Three Dimensional Figures

*Identify properties of three dimensional figures*

Three dimensional shapes have many attributes such as faces, edges andvertices. The flat surfaces of the 3D shapes are called the faces. The line segment where two faces meet is called an edge. Avertexis a point where 3 edges meet.




## The Angle Between a Line and a Plane

*Find the angle between a line and a plane*

In finding the angle between the line and a plane in a three dimensional geometry, we use the right angled triangle. Joining the line to define the angle between the line and the plane that provides the least possible angle. Also, projection of one line to another on the plane is mostly used.

**Example 1**

For the pyramid VABCD with VA=VB=VC=VD=5m, and ABCD a square of side 4cm; find the angle between VA and ABCD.

**Solution**

Calculated by dropping a perpendicular from V to ABCD. This meets ABCD at X, the centre of the square.

So the projection of VA on ABCD is AX. AC= square root of (AB2 +BC2) =square of 42+42 =square root of 32. AX =1/2 square root of 32. cos (1/2 square root 32)/5=0.5657, so VAX is 55.6.

## The Angle Between Two Planes

*Calculate the angle between two planes*

There are infinite possible lines that could be drawn on planes, making different angles with each other. The angle between planes is the angle between lines within those planes, Must be the lines which are at the middle of the plane for non rectangular planes and any other lines for rectangular planes. Then Right angled triangles are used to find the angles between those planes.

**Example 2**

Determine the **angle** between the following **planes**:

$$\pi_1 \equiv 2x - y + z - 1 = 0 \qquad \pi_2 \equiv x + z + 3 = 0$$

$$\vec{n_1} = (2, -1, 1) \qquad \vec{n_2} = (1, 0, 1)$$

$$\alpha(\pi_1, \pi_2) = \alpha(\vec{n_1}, \vec{n_2}) = \arccos \frac{|2 \cdot 1 + (-1) \cdot 0 + 1 \cdot 1|}{\sqrt{2^2 + (-1)^2 + 1^2} \cdot \sqrt{1^2 + 0^2 + 1^2}} = \frac{3}{\sqrt{6} \cdot \sqrt{2}}$$

$$(\widehat{\pi_1, \pi_2}) = \arccos \frac{\sqrt{3}}{2} = 30^\circ$$

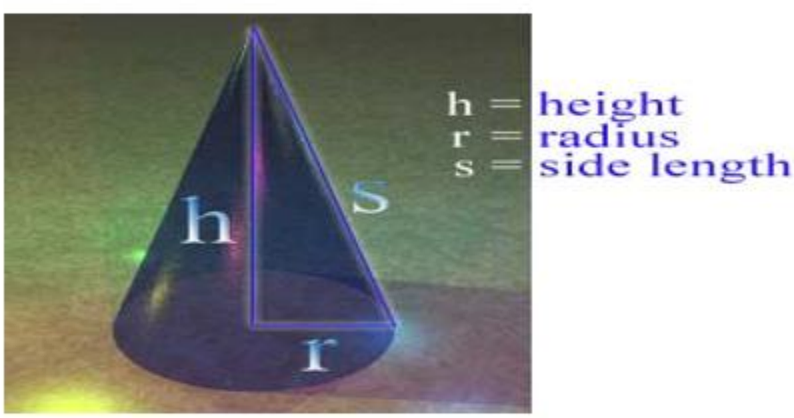
## Surface Area of Three Dimensional Objects

### The Formulae for Calculating the Surface Area of Prisms, Cylinder and Pyramids and Cone

*Derive the formulae for calculating the surface area of prisms, cylinder and pyramids and cone*

Surface Area of a Right Circular Cone

A right circular cone is a cone whose vertex is vertically above the centre of the base of the cone.




$$\text{Area of curved surface} = \frac{1}{2} s \times 2\pi r = \pi rs$$
$$\text{So Area of curved surface} = \pi rs$$

Area of circular base $= \Pi r^2$ (it is an area of a circle)

Therefore the total surface area of a right circular cone $= \Pi r^2 + \Pi rs = \Pi r (r + s)$

$\therefore$ $A = \pi r(r + s)$ where ***r*** is the base radius and ***s*** is the slant height (side length).

**Example 3**

Find the total surface of right circular cone whose slant light is 10cm and whose base radius is 8cm.Use $\Pi r(r + s)$

Solution:

r = 8cm    s=10cm

Surface area = 3.14 x 8 (8 + 10) $cm^2$

= 3.14 x 8 x 18

= $452.16cm^2$

$\therefore$Total surface are = $452.16cm^2$

**Example 4**

Find the total surface area of a cone with diameter 8m and slant height of 10m. Use $\Pi$= 3.14

Solution:

$A = \pi r(r + s)$

d = 8, so r=4

A = 3.14 x 4 (4 + 10)

= 3.14 x 4 x 40

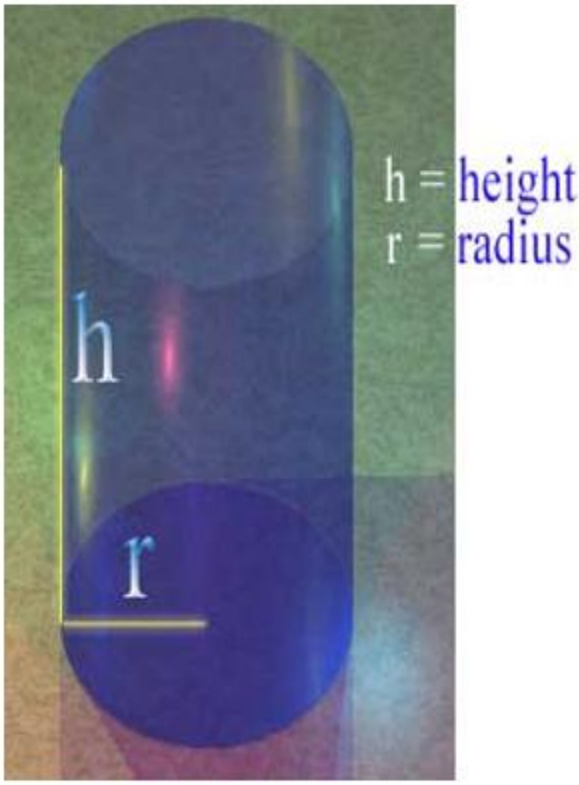
= 175.84 $m^2$

Therefore the total surface area is **175.84 $m^2$**

Surface Area of A Right Cylinder

If you want to know the amount of the covering the surface of a blue band margarine can, then you are finding the surface area of a right cylinder. Total surface area of the can is the sum of the areas of the top and bottom. Circular surfaces plus the area of the curved surface,

Now, consider a right cylinder of radius r and height h.




If the cylinder is opened up, the curved surface flattens out to form a rectangle. The length of the rectangle is 2Πr(the circumference of the circular base) and the width is h (the height of the cylinder).

Total surface area of cylinder:

= Area of curved surface + Area of two bases.

Area of curved surface $= 2\pi rh + 2\pi r^2$

$= 2\pi r(r + h)$

∴ The total surface area of a right cylinder is given by $\boxed{A = 2\pi r(r + h)}$

**Example 5**

Find the total surface area of a cylinder with radius of 3m and height of 10m. Use Π= 3.14

***Solution:***

$A = 2\pi r(r + h)$

$r = 3, \quad h = 10$

Substituting:

$A = 2 \times 3.14 \times 3\ (3 + 10)$

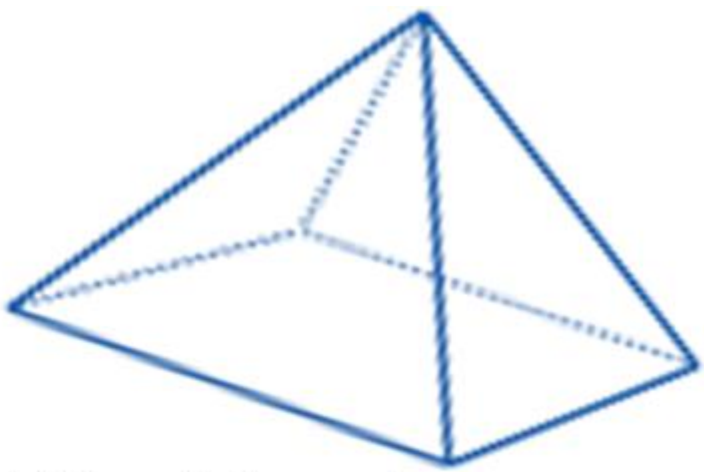
$= 6 \times 3.14 \times 13$

$= 244.92cm^2$

$\therefore$Total surface area is $\mathbf{244.92cm^2}$

***Surface Area of a Right Pyramid***

A right pyramid is one in which the slant edges joining the vertex to the corner of the base are equal



A right pyramid with a square base.

| ***Total surface area = area of lateral surfaces + area of base.*** |
|---|

**Example 6**

A right rectangular pyramid is such that the rectangle is 12cm by 8cm and each slant edge is 12cm. Find the total surface area of the pyramid.

***Solution:***

By Pythagoras a slant edge from the midpoint of the base length to the common vertex of the pyramid is $\sqrt{12^2-6^2}=6\sqrt{3}$ and that from the midpoint of the base width is $\sqrt{12^2-4^2}=8\sqrt{2}$

Area of lateral surfaces = 2 (½ x 12 x$6\sqrt{3}$ ) + 2(½ x 8 x $8\sqrt{2}$)

$= 72\sqrt{3} + 64\sqrt{2}$

= 124. 71 + 90.51

= 215.22cm$^2$

Area of rectangular base = l x w

= 12 x 8cm$^2$

= 96cm$^2$

Total surface area = area of lateral surfaces + area of the base

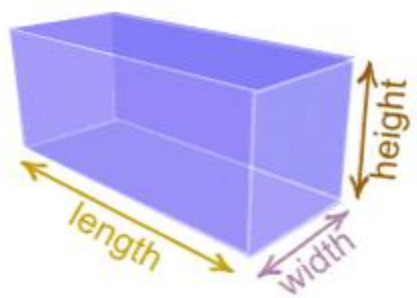
= (215.22 + 96) cm$^2$

= 311.22cm$^2$

***∴Area =311.22cm$^2$***

***Surface Area of a Right Prism***

A full brick or concrete block is an example of a right rectangular prism




A right prism is a prism in which each of the vertical edges is perpendicular to the plane of the base.

The figure above shows a rectangular right prism in which there are 6 faces though only three of them can be seen easily.

Surface Area = Total or sum of the areas of each face.

Generally for any right prism,

***Total surface area = area of lateral surface + area of bases***

**Example 7**

The height of a right prism is 4cm and the perimeter of its base is 30cm. Find the area of its lateral surface.

***Solution:***

Area of lateral surface = perimeter of base x height

= 30 x 4cm$^2$

= ***120cm$^2$***

**Example 8**

Find the total surface area of a rectangular prism 12 by 8 by 6 cm high.

Solution:

Lateral surface area = 6 x 2 (12 + 8) cm$^2$

= 240cm$^2$

Area of base = 2 (12 x 8)

= 192 cm$^2$

Total surface area = (240 + 192) cm$^2$

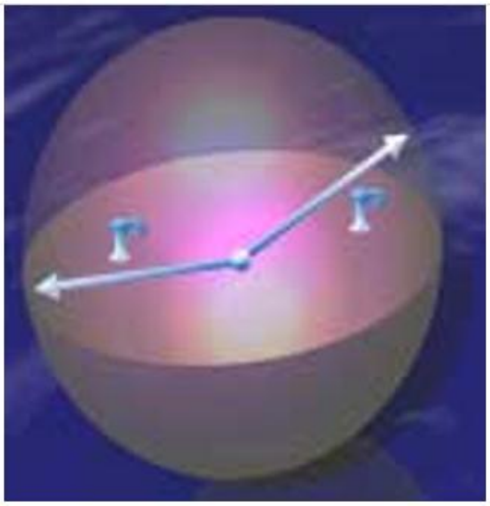
= 432 cm$^2$

∴ ***Total surface area = 432 cm$^2$***

## The Formulae to Calculate the Surface Area of Spheres

*Apply the formulae to calculate the surface area of spheres*

***Surface Area of a Sphere***




The figure above shows a sphere (ball) with radius "r"

The surface area of a sphere is four times the area of circle with the same radius. The area of a circle is $\Pi r^2$. Hence, the surface area of sphere is equal to $4\Pi r^2$.

| |
|---|
| ***Surface area of a sphere = $4\pi r^2$*** |

**Example 9**

Find the surface area of a sphere of radius 5cm. ($\Pi$= 3.14)

***Solution:***

Surface area of sphere = $4\pi r^2$.

= 4 x 3.14 x 5 x 5cm$^2$

$\therefore$The surface area is **314cm$^2$.**

**Example 10**

Find the surface area of a tennis ball, given that its radius is 3.3cm. Use$\Pi$= 3.14 Express your answer to the nearest tenth.

***Solution:***

$A = 4\pi r^2$

So A = 4 x 3.14 x (3.3)2

= (12.56) (10.89)

= 136.7784

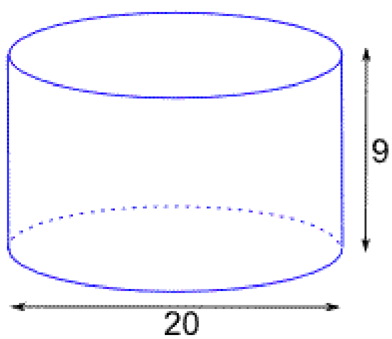
∴The surface area is **136.8cm²**.

## Exercise 1

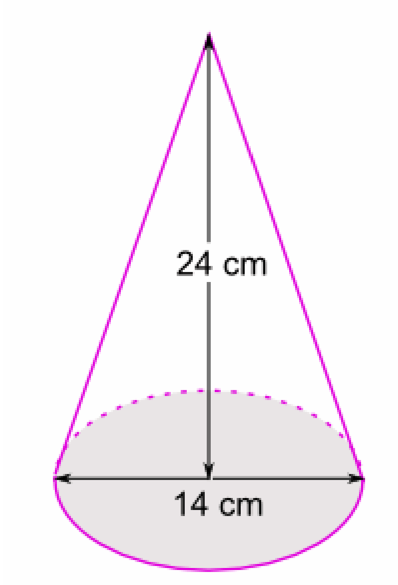
Do the exercise to check your understanding. Use π= 3.14 throughout the exercise.

1.The altitude of a rectangular prism is 4cm and the width and lengths of its base are 2cm and 3cm respectively calculate the total surface area of the prism.

2. The following diagram shows a cylinder of diameter 20 units and height 9 units. What is its curved surface area?




3. The diagram below shows a cone of height 24 cm and base diameter 14 cm. what is its total surface area?




4. The base of a right pyramid is a rectangle 6cm by 8cm and the slant edges are each 9cm long. Calculate its lateral surface area.

5. Find the total surface area of a circular cylinder of diameter 8cm and height 6cm.

6. Taking the earth to be a sphere with radius 6400km, find its surface area.

## Volume of Three Dimensional Objects

## The Formulae for Calculating Volume of Prisms, Cylinders and Pyramids

*Derive the formulae for calculating volume of prisms, cylinders and pyramids*

*VOLUMS OF SOME THREE – DIMENSIONAL FIGURES*

We have seen some formulas for calculating the surface areas of some three dimensional figures.

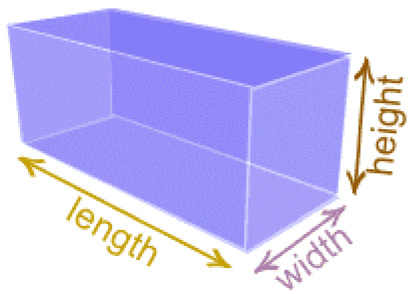
Let us see as well formulas for calculating the volumes of such figures.

-The amount of space that is enclosed by a space figure is called the ***volume***.

The Volume is measured in cubic units, cubic meters ($m^3$), Cubic centimeters ($cm^3$) etc.

When we find (calculate) the volume of a space figure or solid, we are finding the number of cubic units enclosed by the given spaces figure.

***(a) Volume of a Right Prism***




The figure above shows a right rectangular prism. Let *h* be height, ***w*** width and $l$ the length of the prism.

Then the Volume of the prism is given by***:*** *V = Base area x height=* $l \times W \times h$

Generally, volume of any right prism is equal to the product of the area of the base and the height V = Base area x height.

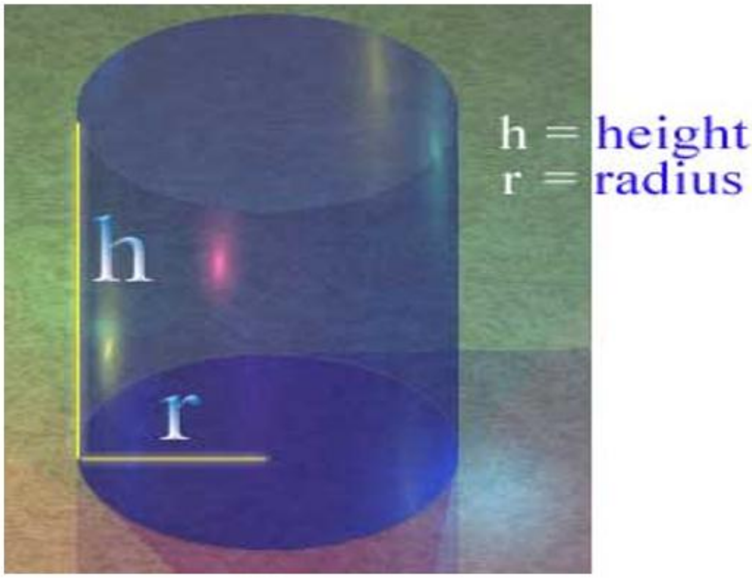
$\therefore$ ***Volume (V) = Base area x height***

Or

$V = B_a \times H$ Where $V = volume$ , $B_a$ = base area and $H$ = height

***(b) Volume of a Right Cylinder***

Consider a right circular cylinder with radius" r "and height h as shown below.




The volume of a right circular cylinder is equal to the product of the area of the base and the height.

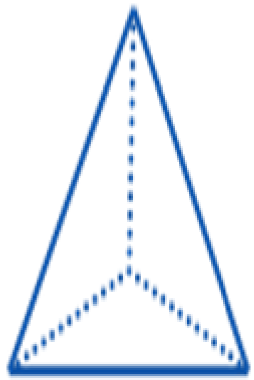
If V is volume, A is area of the base and h is the height,

Then Volume = Area of base x height

Or $V = \pi r^2$ x h where $\pi r^2$ is the base area (a circular bases area) and h is the height of the cylinder.

$$\therefore \boxed{V = \pi r^2 h}$$

***(c) Volume of a Pyramid***



Generally, the volume of a pyramid is one – third the product of its altitude (height) and its base area.

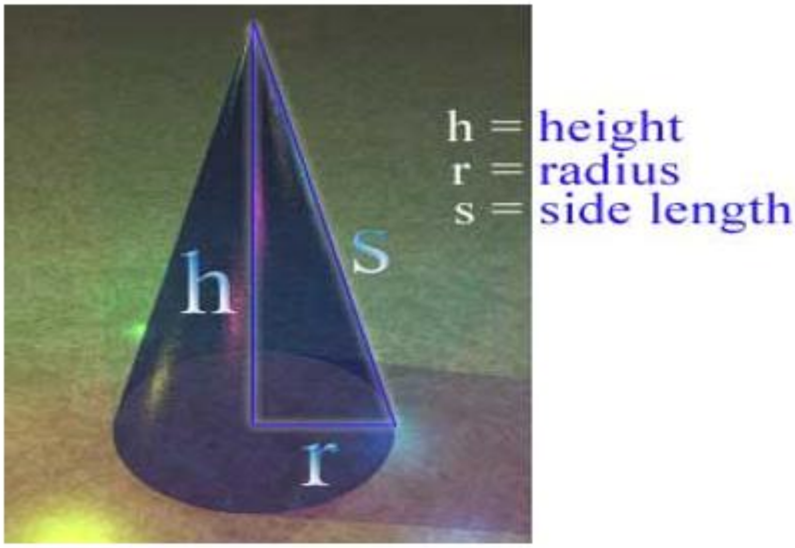
If h is the perpendicular distance from the vertex of the pyramid to its base then,

Volume of the pyramid $= \frac{1}{3}$(base area x height)

$$V = \frac{1}{3}(\text{base area x height})$$

***(d)Volume of a Cone***

Consider a cone of radius "r" and altitude h as shown below.



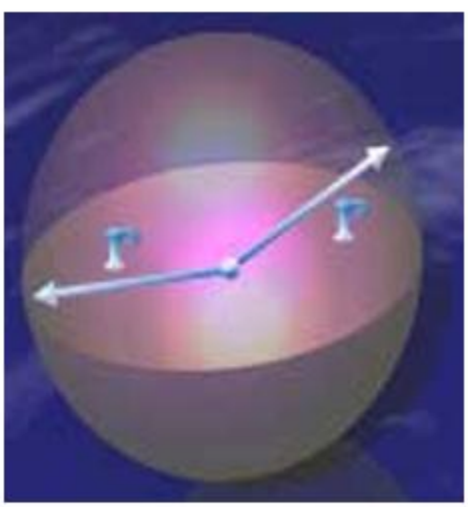

A base of a circular cone can be considered to be a regular polygon with very many (infinitive) numbers of sides.

Like a pyramid, Volume of a circular cone is one – third the product of its altitude and its base area.

If the base is of radius "r", then the base area is $\pi r^2$ and the volume becomes $\frac{1}{3}\pi r^2 h$

$$\therefore \boxed{\textit{The volume of the cone}(V) = \frac{1}{3}\pi r^2 h}$$

***(e) Volume of a Sphere***



The figure above shows a sphere of radius r, if the sphere can be put inside a cylinder of the same radius" r", then the height h = 2r.

It follows that Volume of Sphere is given by Volume $=\frac{4}{3}\pi r^3$

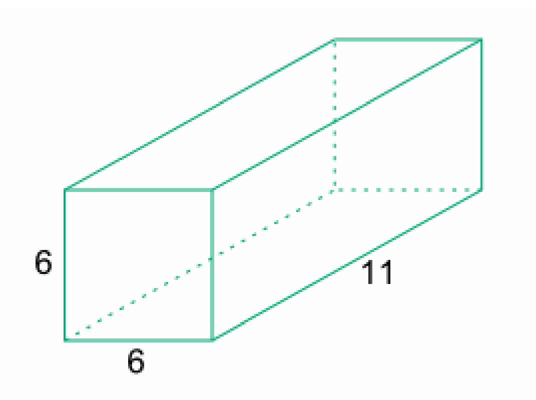
∴ $$\textit{Volume of sphere} = \frac{4}{3}\pi r^3$$

## The Formulae to Calculate the Volume of Cylinders, Pyramids and Cones

*Apply the formulae to calculate the volume of cylinders, pyramids and cones*

**Example 11**

Find the volume of the prism shown below, given that the dimensions are in meters (m)




***Solution:***

The base of the prism is a rectangle.

Area of base = $l \times w = 11 \times 6 = 66\ m^2$

h= 6m

v= Base area x height (h)

$v = 66m^2 x\ 6m = 396m^3$

$\therefore \textit{Volume of prism} = 396m^3$

**Example 12**

Calculate the volume of a rectangular prism whose base is 8cm by 5cm and whose height is 10cm.

***Solution:***

Volume = Area of Base x height

= (8 x 5) x $10cm^3$

= 400cm3

∴***The volume is $400cm^3$.***

**Example 13**

Calculate the radius of a right circular cylinder of volume $1570m^3$ and height 20m. Use π=3.14

***Solution:***

$V = \pi r^2 x\ h$

So $1570 = 3.14 \times r^2 \times 20$

$r^2 = \frac{1570}{3.14 \times 20}$

$r^2 = 25$

r = 5m

∴***The radius is 5m.***

**Example 14**

A pipe made of metal 1cm thick, has an external (outside) radius of 6cm. Find the volume of metal used in making 4m of pipe. Use π=3.14

***Solution:***

Let R be the external radius and r the inner radius, and h the height,

Volume of pipe = area of cross section x height

$= (\pi R^2 - \pi r^2)\ h$

$= \pi R^2 h - \pi r^2 h$

$V = \pi\ (R^2 - r^2)h$

$= \pi(R + r)(R - r)\ h$

R = 6cm, r = 5cm, h= 4000cm

V= 3.14 x 11 x 1 x $400m^3$

= 13816 $cm^3$

∴***Volume of metal is $13816cm^3$***

**Example 15**

Find the volume of a pyramid with rectangular base with length 6m and width 4m if the height of the pyramid is 10m.

**Solution**

Base area = length x width = $6 \times 4m^2 = 24m^2$

From $V = \frac{1}{3}$(base area x height), h= 10m

$V = = \frac{1}{3}(24m^2 \times 10m) = 80m^3$

***∴The volume is 80m³***

## Example 16

Calculate the volume of a square pyramid whose altitude is 10cm and length of side of base is 6cm.

***Solution:***

$V = \frac{1}{3}$(base area x height) $= \frac{1}{3}(6 \times 6 \times 10)\ cm^3 = 12 \times 10\ cm^3$

$= 120cm^3$

***∴The volume is 120cm³***

## Example 17

Calculate the volume of a cone having base radius 10cm and altitude 12 cm Use π=3.14

***Solution:***

Volume $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times 3.14 \times 102 \times 12cm^3$

$= 1256cm^3$

***∴The volume is 1256cm³.***

## Example 18

Find the volume of a sphere whose radius is 10cm. (Take π= 3.14).

***Solution***: Volume $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times 3.14 \times 10 \times 10 \times 10$

$= 4.1867 \times 10^3$

$= 4186.7cm^3$

***∴The volume is 4186.7cm³***

### Example 19

The volume of a spherical tank is $268m^3$. Calculate the radius of the tank. ($\pi$= 3.14)

***Solution***: Volume $= \frac{4}{3}\pi r^3$

$$r^3 = \frac{3}{4\pi}V$$

$$r^3 = \frac{3 \times 268}{3 \times 3.14}$$

$$= 64.0$$

$$r = \sqrt[3]{64} = 4m$$

$\therefore$ ***The radius of the tank is 4m.***

### Example 20

Find the volume of rubber in a hollow spherical ball with inner diameter 14cm and outer diameter 16cm. (Take $\pi$ = 3.14)

***Solution:***

Let inner radius be r and outer radius be R.

Then R = 8cm and r = 7cm

Volume of spherical shell = Volume of outer sphere − Volume of inner sphere.

$$V = \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi\,(R^3 - r^3)$$

But R = 8 and r = 7

$$V = \frac{4}{3}\pi(R^3 - r^3) = \frac{4}{3}\pi(8^3 - 7^3) = \frac{4}{3} \times 3.14 \times 169$$

$$= 707.55cm^3$$

$\therefore$***The volume of rubber is*** $707.55cm^3$

### Exercise 2

Answer the following questions and (use $\pi$ = 3.14)

1. What is the volume of a right prism whose base is a regular hexagon ( n = 6) with a side of the base 4cm long and the height of the prism.

2. Find the volume of a cylinder whose diameter is 28cm and whose height is 12cm.

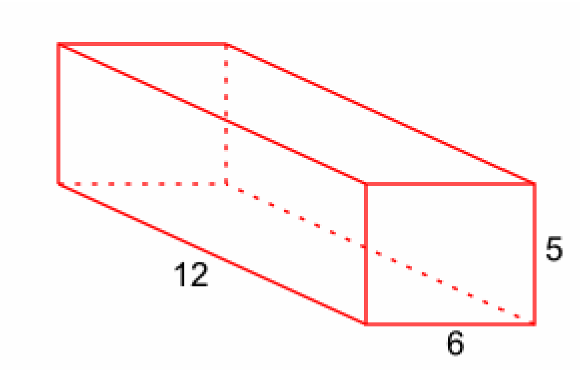
3. Find the volume of a square pyramid whose height is 24cm and slant edge 25cm each.

4. The slant height of a cone is 20cm and the radius of its base is 12cm. Find its volume in terms of $\pi$.

5. The volume of a sphere is $827cm^3$. Find its radius.

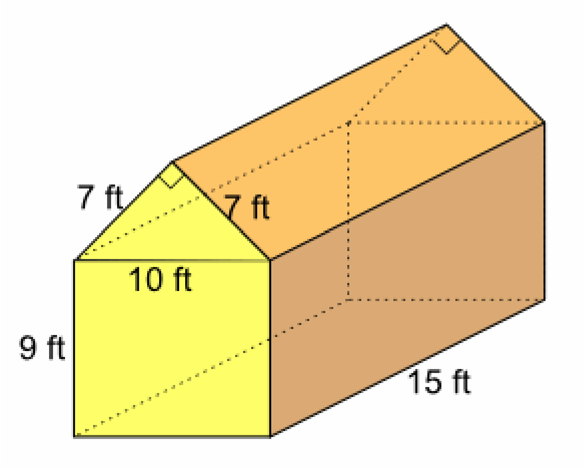
6. A cylinder and sphere have the same volume. If the radius of the sphere is 5cm and radius of the cylinder is 3cm, Calculate height of the cylinder.

7.

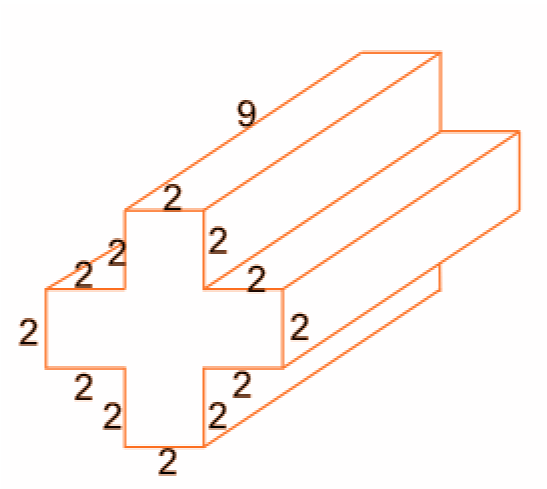



Find the surface area of this rectangular prism (cuboid)

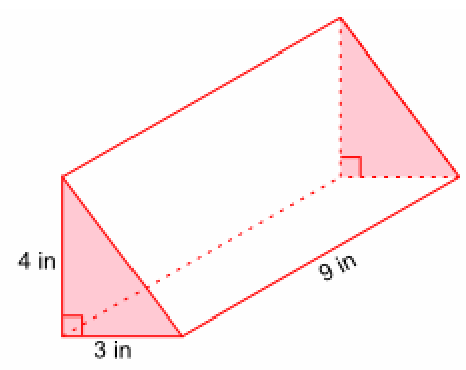
8.The diagram shows a barn. What is the volume of the barn? (The length of the hypotenuse in the right triangle is rounded to the nearest foot.)



9.What is the volume of this prism?




The diagram shows a prism whose cross-section is a right triangle. What is the volume of the prism?




***Summary of the topic***

Here are the important formulas you have covered under the section on surface areas summarized.

Surface area of:

1. Right circular cone.
   Curved surface = $\pi rl$

   Circular base = $\pi r^2$

   Total surface area, $A = \pi rl + \pi r^2$
   Or $A = \pi r(r + l)$
2. Right cylinder
   Curved surface $= 2\pi rh$
   Two bases $= 2\pi r^2$
   Total surface area, $A = 2\pi r^2 + 2\pi rh$
   Or $A = 2\pi r[r + h]$

3. Right Pyramid
   Total surface area = area of lateral surface + area of base.

4. Right Prism
   Area of lateral surface = perimeter of base x height
   Total surface area = area of lateral surface + area of base

5. Sphere
   Surface area = $4\pi r^2$.

You can now have a summary of the important formulas for calculating volume of some three dimensional figures as follows:-

1. Prism: V = base area x height

2. Cylinder: V = area of circular base x height $= \pi r^2 h$
3. Cone: $V = \frac{1}{3}$ (area of circular base x height) $= \frac{1}{3}\pi r^2 h$

4. Pyramid: $V = \frac{1}{3}$ (base area x height)